Arzela-Ascoli Theorem

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1 Introduction

1.1 Preliminaries on Metric Spaces

A metric space is any set of elements that are endowed with a metric, a metric being a function defining the distance between any two elements in the space. While the space of real numbers, $\mathbb{R}$, has the well-known standard Euclidean notion of distance, metric spaces are useful because they generalize that notion of distance to other sets. Other common examples of metric spaces include polynomials, infinitely long sequences, and functions. Note that $\mathbb{R}$ is one of the simplest examples of a metric space, where the metric is the standard Euclidean distance, this being the absolute value of the difference between any two real numbers.

For any metric space, the metric function maps each pair of elements in the space to positive real values, and must satisfy certain properties –

- The distance between two elements is 0 if and only if the elements are the same.
- The distance between elements A and B in the metric space is the same as the distance between the elements B and A.
- The metric satisfies the triangle inequality familiar from the real numbers.

1.2 Compactness

Compactness is a characterization of some metric spaces, and compact metric spaces satisfy a number of elegant and useful properties. The notion of compactness is essentially equivalent for metric spaces and subsets of metric spaces, in the sense that it describes the same property. There is a minor technical difference between the two, this being that when determining if a subset of a metric space is compact, one must consider the relative metric restricted to said subset. This does not affect the intuition of compactness and is out of the scope of this introduction (see Chapters 41 and 42 of [1] for a detailed treatment of this).

The Heine-Borel theorem nicely characterizes that a subset of the real numbers is compact if and only if it is closed and bounded, thus $\mathbb{R}$ itself is not compact because it is unbounded, but $[0, 1] \subset \mathbb{R}$ is compact. Three useful results regarding compact metric spaces are –

- One familiar theorem from calculus, the "extreme value theorem," can be generalized using compactness, namely that all continuous real-valued functions on compact metric spaces attain a maximum and minimum on the metric space.
- Another familiar theorem from real analysis, the Bolzano-Weierstrass theorem, generalizes to compact metric spaces, namely that all sequences of elements in a compact metric space have some convergent subsequence, irrespective of whether the original sequence converges or not.

- For any continuous function f defined on a compact metric space space M_1, its range $f(M_1)$ is also compact, and it is also uniformly continuous on M_1.

1.3 Space of Continuous, Bounded, Real-Valued Functions on $[0, 1]$

We consider in the Arzela-Ascoli theorem a specific metric space, that of continuous, bounded, real-valued functions on the interval $[0, 1]$. This metric space is often denoted $\mathcal{C}_b([0, 1])$, however henceforth we denote it as $\mathcal{C}$ for brevity. The metric on this space, a function $d : \mathcal{C} \times \mathcal{C} \to [0, \infty)$, is defined as follows for all functions $f, g \in \mathcal{C}$:

$$d(f, g) := \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}$$

With this definition, it can be proven that $(\mathcal{C}, d)$ is a metric space, the proof of which is outside of the scope of this introduction, but involves showing that d satisfies all of the properties of a metric as outlined in Section 1.1 (see Example 16.7 in [2] for a detailed treatment of this).

1.4 Motivation for Arzela-Ascoli

As mentioned in Section 1.2, the Heine-Borel theorem gives us an "easy" sufficient condition to determine if subsets of real coordinate spaces $X \subset \mathbb{R}^k$ are compact, namely that if X is closed and bounded then it is compact. However, this simple condition does not generalize to arbitrary metric spaces, and we must check a more arduous sufficient condition such as one of the following –

- Every open cover of X has a finite subcover.
- Every sequence of elements in X has a convergent subsequence.
- X is totally bounded and complete.

For $\mathcal{C}$, the Arzela-Ascoli theorem provides a similarly "easy" sufficient condition to establish that $\phi \subset \mathcal{C}$ is compact, functioning in a similar capacity as the Heine-Borel theorem does for $\mathbb{R}^k$, namely that if $\phi \subset \mathcal{C}$ is uniformly bounded and equicontinuous, then the closure of ϕ, denoted $\overline{\phi}$, is compact.

For instance, from the Arzela-Ascoli theorem it directly follows that a finite set of constant functions is compact. More formally, for a fixed $k \in \mathbb{N}$, the closure of the set $\phi \subset \mathcal{C}$ containing the functions satisfying for all $x \in [0, 1]$, $f(x) = n$ for some $n \in \{1, 2, \cdots, k\}$, is compact by the Arzela-Ascoli theorem. Once the definitions of the conditions for the Arzela-Ascoli theorem are introduced in Section 2.1, it should be obvious that $\overline{\phi}$ is compact, because ϕ clearly is uniformly bounded and equicontinuous. Using one of the three more general characterizations of compactness from above instead of the Arzela-Ascoli theorem, it would take much more effort to establish that $\overline{\phi}$ is compact.

2 Proof of Arzela-Ascoli Theorem

2.1 Statement of Theorem

We begin with some definitions of the sufficient conditions of the Arzela-Ascoli theorem. Let $\mathcal{C}$ denote the space of continuous, bounded real-valued functions on the interval $[0, 1]$ as described in Section 1.3. Then, we define the following properties of a subset $\phi \subset \mathcal{C}$.

Definition 1. *A subset $\phi \subset \mathcal{C}$ is uniformly bounded if there exists $B \in \mathbb{R}$ such that, for all $x \in [0, 1]$ and all $f \in \phi$, $|f(x)| \leq B$.*

Note that this $B \in \mathbb{R}$ bounds all $f \in \phi$ for all values in their domain $x \in [0, 1]$, which is stronger than the typical definition of boundedness.

Definition 2. *A subset $\phi \subset \mathcal{C}$ is equicontinuous if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \phi$ and all $x, y \in [0, 1]$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.*

Note that this $\delta > 0$ must satisfy that the outputs of the function are of distance less than ε not only for each function $f \in \phi$ but also for all $x, y \in [0, 1]$, making equicontinuity a stronger condition than uniform continuity.

Finally, for any $\phi \subset \mathcal{C}$, let $\overline{\phi} \subset \mathcal{C}$ denote the set of all limit points of (also known as the closure) ϕ. Then, we can succintly state the Arzela-Ascoli theorem as follows –

Theorem 1. *If a subset $\phi \subset \mathcal{C}$ is uniformly bounded and equicontinuous, then $\overline{\phi}$ is compact.*

2.2 Outline of Proof

The bulk of the proof is dedicated to showing that $\phi \subset \mathcal{C}$ is totally bounded, thus it will be helpful to begin by defining this property.

Definition 3. *A subset of a metric space $X \subseteq (M, d)$ is totally bounded if for all $\varepsilon > 0$, there exists a finite $x_1, \cdots, x_n \in X$ such that $X \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)$ with respect to the relative metric d_X.*

Given $\gamma > 0$, we will show that ϕ can be totally bounded by finitely many balls of radius γ. To do so, we will approximate each of the (potentially infinitely many) functions $\varphi \in \phi$ with a finite number of piecewise linear functions ψ. Then, we will bound these linear functions such that all functions $\varphi \in \phi$ are contained within at least one ball of radius γ about each of these finitely many ψ, in other words that ϕ is totally bounded. Finally, we will combine the fact that ϕ is totally bounded with the fact that $\overline{\phi}$ is complete to conclude that $\overline{\phi}$ is compact.

2.3 Proof

We first show that $\phi \subset \mathcal{C}$ is totally bounded. Let $\gamma > 0$ be given, and define $\varepsilon := \frac{\gamma}{5}$, then our goal is to show that ϕ can be covered by finitely many balls of radius γ. Because ϕ is uniformly bounded, we know that there exists some $B \in \mathbb{R}$ such that for all $x \in [0, 1]$ and all $f \in \phi$, $|f(x)| \leq B$. We also know that because ϕ is equicontinuous, there exists $\delta > 0$ such that for all $f \in \phi$ and all $x, y \in [0, 1]$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Next, we divide $[0, 1]$ into finitely many subintervals, each of length less than δ. Denote the points of subdivision of $[0, 1]$ as $0 = x_0 < x_1 < \cdots < x_n = 1$. Also, we divide the interval $[-B, B]$ into finitely many subintervals, each of length less than ε. We denote these points of subdivision of $[-B, B]$ as $-B = y_0 < y_1 < \cdots < y_p = B$.

Combining these subdivisions, the rectangle $[0, 1] \times [-B, B]$ is divided into np smaller rectangles, each with width at most δ and height at most ε. For each $\varphi \in \phi$, define a continuous piecewise linear approximation $\psi : [0, 1] \to [-B, B]$. For all $k \in \{0, 1, \ldots, n\}$, there exists by construction some $\ell \in \{0, 1, \ldots, p\}$ such that $|y_\ell - \varphi(x_k)| < \varepsilon$. Given this, for all $k \in \{0, 1, \ldots, n\}$, define $\psi(x_k) := y_\ell$. Note that by construction, $|\psi(x_k) - \varphi(x_k)| = |y_\ell - \varphi(x_k)| < \varepsilon$.

Then, for any $x \in [0, 1]$, we know that by definition of a subdivision that there exists some $k \in \{0, 1, \ldots, n - 1\}$ such that $x_k < x < x_{k+1}$. Define

$$\psi(x) := \psi(x_k) + \frac{\psi(x_{k+1}) - \psi(x_k)}{x_{k+1} - x_k}(x - x_k)$$

as the linear approximation of this $\varphi \in \phi$.

Now, let $k \in \{0, 1, \ldots, n - 1\}$. Recall that by construction, $|x_{k+1} - x_k| < \delta$. Because ϕ is equicontinuous and $\varphi \in \phi$, we know that $|\varphi(x_k) - \varphi(x_{k+1})| < \varepsilon$. We also showed above that $|\psi(x_k) - \varphi(x_k)| < \varepsilon$. Thus, apply the triangle inequality and these bounds to see that

$$|\psi(x_k) - \psi(x_{k+1})| \leq |\psi(x_k) - \varphi(x_k)| + |\varphi(x_k) - \varphi(x_{k+1})| + |\varphi(x_{k+1}) - \psi(x_{k+1})| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

From here, consider the same $k \in \{0, 1, \ldots, n-1\}$ and some $x \in [x_k, x_{k+1}]$. By construction of ψ as a piecewise linear function, we know that on each linear "piece" of the function (in other words, each interval $[x_k, x_{k+1}]$), it is strictly decreasing, constant, or strictly increasing. This implies that for $x \in [x_k, x_{k+1}]$, $|\psi(x_k) - \psi(x)| \leq |\psi(x_k) - \psi(x_{k+1})| < 3\varepsilon$.

Now, let $x \in [0, 1]$ and consider the quantity $|\varphi(x) - \psi(x)|$. Similar to before, let $k \in \{0, 1, \ldots, n-1\}$ be such that $x \in [x_k, x_{k+1}]$. Because $|x - x_k| < \delta$ by construction and ϕ is equicontinuous, we know that $|\varphi(x) - \varphi(x_k)| < \varepsilon$. Also, recall from above that $|\varphi(x_k) - \psi(x_k)| < \varepsilon$ by construction of ψ, and we just showed that $|\psi(x_k) - \psi(x)| < 3\varepsilon$. Then, the triangle inequality gives us that

$$|\varphi(x) - \psi(x)| \leq |\varphi(x) - \varphi(x_k)| + |\varphi(x_k) - \psi(x_k)| + |\psi(x_k) - \psi(x)| < \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon \quad (1)$$

Consider again the subset $\phi \subset \mathcal{C}$. Note that there exist only np vertices of the smaller rectangles of $[0, 1] \times [-B, B]$ and the functions ψ are defined in such a way that they are piecewise linear. For each of the n subdivision points $x_0, \ldots, x_n$ that subdivide $[0,1]$, there exist at most p choices of subdivision points $y_0, \ldots, y_p$ that subdivide $[-B, B]$ to which any $\psi : [0,1] \to [-B, B]$ could map to. For any two functions in ϕ, if the approximation ψ maps the same subdivision points $\psi(x_k) := y_\ell$ for each $k \in \{0, 1, \ldots, n\}$, then their approximations ψ are the same. Thus, there exist no more than p^n possible approximations ψ for all of the functions in ϕ.

Now, let $1 \leq P \leq p^n$ be the number of linear approximations ψ of all functions in ϕ. We showed that there exist finitely many $\psi_0, \cdots, \psi_P$, each of which is an element of $\mathcal{C}$. Recall from Section 1.3 that for $\varphi \in \phi$ and $k \in \{0, \ldots, P\}$, we know that $d(\psi_k, \varphi) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}$. By equation (1), we know that there exists some $k \in \{0, \ldots, P\}$ such that the quantity $d(\psi_k, \varphi) < 5\varepsilon$. In other words,

$$\phi \subseteq B_{5\varepsilon}(\psi_0) \cup \cdots \cup B_{5\varepsilon}(\psi_P) = B_\gamma(\psi_0) \cup \cdots \cup B_\gamma(\psi_P)$$

thus we have established that ϕ is totally bounded.

Now that we have shown that ϕ is totally bounded, we must show that this implies that $\overline{\phi}$ is totally bounded. Let $\varepsilon > 0$, then because ϕ is totally bounded, there exists some finite $x_1, \cdots, x_n$ such that $\phi \subseteq B_{\frac{\varepsilon}{2}}(x_1) \cup \cdots \cup B_{\frac{\varepsilon}{2}}(x_n)$. Now, let $\varphi \in \overline{\phi}$. We know by definition of the closure that there exists a sequence $\{\varphi_n\} \subseteq \phi$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. By definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(\varphi_n, \varphi) < \frac{\varepsilon}{2}$. Also, we know that because $\varphi_n \in \phi$, there exists $k \in [1, n]$ such that $\varphi_n \in B_{\frac{\varepsilon}{2}}(x_k)$ which implies that $d(x_k, \varphi_n) < \frac{\varepsilon}{2}$. Then, by the triangle inequality,

$$d(x_k, \varphi) \leq d(x_k, \varphi_n) + d(\varphi_n, \varphi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, we have shown that the union $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)$ totally bounds $\overline{\phi}$, because all functions $\varphi \in \overline{\phi}$ are contained in one of these balls of radius ε.

Note that $\mathcal{C}$ is complete (see Example 17.16 in [2] when $D = [0,1] \subseteq \mathbb{R}$). By definition of a complete metric space, every Cauchy sequence $\{\varphi_n\} \subseteq \mathcal{C}$ converges to some $\varphi \in \mathcal{C}$. We now want to show that $\overline{\phi}$ is also complete.

Let $\{\varphi_n\} \subseteq \overline{\phi} \subseteq \mathcal{C}$ be a Cauchy sequence in $\overline{\phi}$. Then, because $\mathcal{C}$ is complete, we know that there exists some $\varphi \in \mathcal{C}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. Note that by construction φ is a limit point of $\overline{\phi}$, and by definition of the closure $\varphi \in \overline{\overline{\phi}}$. However, because $\overline{\phi} \subset \mathcal{C}$ is closed (see Proposition 6.11(e) of [2]), $\overline{\overline{\phi}} = \overline{\phi}$, therefore $\varphi \in \overline{\phi}$. Thus, we have shown that all Cauchy sequences $\{\varphi_n\} \subseteq \overline{\phi}$ converge to some $\varphi \in \overline{\phi}$, in other words that $\overline{\phi}$ is complete.

We showed earlier that ϕ is totally bounded which implied that $\overline{\phi}$ is totally bounded. Because $\overline{\phi}$ is totally bounded and complete, it is compact (see Theorem 14.12 proven on page 205 of [3]). $\square$

2.4 Examples of Non-Compact Sets of Continuous Functions on $[0, 1]$

Our goal for this section is to give examples of subsets of $\mathcal{C}$ that are either not equicontinuous or not uniformly bounded and show that their closures are not compact to demonstrate the necessity of both sufficient conditions of the Arzela-Ascoli theorem.

Example 1. We propose a subset of $\mathcal{C}$ that is equicontinuous but not uniformly bounded, and show that it is not compact without using the Arzela-Ascoli theorem.

Consider the subset $\phi := \{\varphi_n\}_{n=1}^{\infty} = \{n \mid n \in \mathbb{N}\} \subseteq \mathcal{C}$ where for all $\varphi_n \in \phi$, $\varphi_n : [0, 1] \to \mathbb{R}$. This set of constant functions is clearly equicontinuous, because given $\varphi_n \in \phi$, for all $x, y \in [0, 1]$ we have $\varphi_n(x) = n = \varphi_n(y)$, which implies that for all $\varepsilon > 0$, $|\varphi_n(x) - \varphi_n(y)| = 0 < \varepsilon$. Because this holds for all $x, y \in [0, 1]$, $\delta = 1$ satisfies the equicontinuity condition.

However, this set ϕ is not uniformly bounded. Suppose for contradiction that there exists $B \in \mathbb{R}$ that uniformly bounds ϕ, then by the Archimedean property of the reals there exists some $N \in \mathbb{N}$ such that $N > B$. Clearly, by construction, $\varphi_N(x) \in \phi$, and for all $x \in [0, 1]$ we have that $|\varphi_N(x)| = N > B$, contradicting the fact that B uniformly bounds ϕ.

Finally, $\overline{\phi}$ is not compact. Consider the sequence $\{\varphi_n\} \subseteq \overline{\phi}$, we claim that there is no convergent subsequence. Suppose for contradiction that there exists some $\{n_k\}$ such that $\{\varphi_{n_k}\}$ converges. Then, the sequence must be Cauchy, in other words for all $0 < \varepsilon < 1$, there exists some $K \in \mathbb{N}$ such that for all $i, j \geq K$, $d(\varphi_{n_i}, \varphi_{n_j}) < \varepsilon$. However, clearly, this is a contradiction because by construction $d(\varphi_{n_i}, \varphi_{n_j}) = \sup\{|\varphi_{n_i}(x) - \varphi_{n_j}(x)| \mid x \in [0, 1]\} \geq 1 \not< \varepsilon$. Thus, there exists no convergent subsequence of $\{\varphi_n\} \subseteq \overline{\phi}$, which implies that $\overline{\phi}$ is not compact (see Theorem 43.5 of [1]).

Example 2. We now propose a subset of $\mathcal{C}$ that is uniformly bounded but not equicontinuous, and show that it is not compact without using the Arzela-Ascoli theorem.

Consider the subset $\phi := \{\varphi_n\}_{n=1}^{\infty} = \{\sin(nx) \mid n \in \mathbb{N}\} \subseteq \mathcal{C}$, where for all $\varphi_n \in \phi$, $\varphi_n : [0, 1] \to \mathbb{R}$. This sequence is uniformly bounded by 1, because when $B = 1$, for all $\varphi_n \in \phi$ and for all $x \in [0, 1]$, $|\varphi_n(x)| \leq 1$ by definition of the sine function.

However, ϕ is not equicontinuous. Let $0 < \varepsilon < 2$ and suppose for contradiction that there exists some $\delta > 0$ such that for all $\varphi_n \in \phi$ and $x, y \in [0, 1]$, $|x - y| < \delta$ implies that $|\varphi_n(x) - \varphi_n(y)| < \varepsilon$. However, we know for $n \in \mathbb{N}$, the period of φ_n is $\frac{2\pi}{n}$. Thus, for $n > \frac{\delta}{2\pi}$, we see that the period of φ_n is less than δ. Thus, for all $z \in [0, 1 - \delta)$, an interval of the form $(z, z + \delta)$ attains both the maximum and minimum of φ_n (for this specific $\varphi_n := \sin(nx)$, the maximum and minimum are 1 and -1 respectively). In other words, for a given $z \in [0, 1 - \delta)$, because $\varepsilon < 2$, we can find two points $x, y \in (z, z + \delta)$ such that $|x - y| < \delta$ but $|\varphi_n(x) - \varphi_n(y)| = 2 \not< \varepsilon$, contradicting the equicontinuity of ϕ.

Finally, we claim that $\overline{\phi}$ is not compact. Consider the sequence $\{\varphi_n\} \subseteq \overline{\phi}$, we claim that there is no convergent subsequence. Suppose for contradiction that there exists some strictly increasing $\{n_k\}$

such that $\{\varphi_{n_k}\}$ is convergent. Then, $\{\varphi_{n_k}\}$ would be Cauchy, which implies that for all $0 < \varepsilon < 1$, there exists some $K \in \mathbb{N}$ such that for all $i, j \geq K$, $d(\varphi_{n_i}, \varphi_{n_j}) = \sup\{|\varphi_{n_i}(x) - \varphi_{n_j}(x)| \mid x \in [0,1]\} < \varepsilon$. Fix $i \geq K$, and denote the period of $\varphi_{n_i}(x)$ as $P = \frac{2\pi}{n_i}$.

Because $\{n_k\}$ is strictly increasing and unbounded, there exists $j \geq K$ such that $n_j > 2n_i$. Then, for this j, consider the the period of $\varphi_{n_j}(x)$,

$$\text{period of } \sin(n_j x) = \frac{2\pi}{n_j} < \frac{\pi}{n_i} = \frac{P}{2}$$

When $x = 0$, note that $\varphi_{n_i}(x) = 0 = \varphi_{n_j}(x)$. By definition of the period of φ_{n_j}, there exists some $y \in (0, \frac{P}{2}) \subset [0,1]$ such that $\varphi_{n_j}(y) = -1$. However, at this $y \in (0, \frac{P}{2})$, φ_{n_i} has not yet completed half of its period; thus $\varphi_{n_i}(y) > 0$. This implies that $|\varphi_{n_j}(y) - \varphi_{n_i}(y)| > 1$ for some $y \in [0,1]$, and thus by definition of the supremum,

$$\sup\{|\varphi_{n_i}(x) - \varphi_{n_j}(x)| \mid x \in [0,1]\} \geq |\varphi_{n_j}(y) - \varphi_{n_i}(y)| > 1 \not< \varepsilon$$

which contradicts the convergence of the subsequence $\{\underline{\varphi}_{n_k}\}$. Thus, this sequence in $\{\varphi_n\} \subseteq \overline{\phi}$ does not have a convergent subsequence, which implies that $\overline{\phi}$ is not compact (see Theorem 43.5 of [1]).

References

[1] Richard Johnsonbaugh and W. E. Pfaffenberger. Foundations of mathematical analysis. Dover Publications, Inc., 2010.

[2] Wilson A. Sutherland. Introduction to metric and topological spaces. Oxford Mathematics, 2009.

[3] Seymour Lipschutz. Schaum's outline of theory and problems of general topology. McGraw Hill, 1965.